A LEVEL Cambridge Topical Past Papers

FURTHER MATHEMATICS

9231 P1,P2

2020 — 2025

1 - (9231/11_Summer_2020_Q2) ANSWER

The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^2, β^2, γ^2. [3]

(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

2 - (9231/12_Summer_2020_Q2) ANSWER

The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^2, β^2, γ^2. [3]

(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

3 - (9231/13_Summer_2020_Q1) ANSWER

The cubic equation $7x^3 + 3x^2 + 5x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are α^{-1}, β^{-1}, γ^{-1}. [3]

(b) Find the value of $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$. [2]

(c) Find the value of $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$. [2]

4 - (9231/11_Winter_2020_Q3) ANSWER

The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^3, \beta^3, \gamma^3$. [3]

(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

5 - (9231/12_Winter_2020_Q1) ANSWER

The cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c and d are constants, has roots α, β, γ. It is given that $\alpha\beta\gamma = -1$.

(a) State the value of d. [1]

(b) Find a cubic equation, with coefficients in terms of b and c, whose roots are $\alpha + 1$, $\beta + 1$, $\gamma + 1$. [3]

(c) Given also that $\gamma + 1 = -\alpha - 1$, deduce that $(c - 2b + 3)(b - 3) = b - c$. [4]

6 - (9231/13_Winter_2020_Q3) ANSWER

The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^3, \beta^3, \gamma^3$. [3]

(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

7 - (9231/11_Summer_2021_Q3) ANSWER

The equation $x^4 - 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^3, \beta^3, \gamma^3, \delta^3$ and state the value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$. [4]

(b) Find the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3}$. [3]

(c) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

8 - (9231/12_Summer_2021_Q3) ANSWER

The equation $x^4 - 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^3, \beta^3, \gamma^3, \delta^3$ and state the value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$. [4]

(b) Find the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3}$. [3]

(c) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

9 - (9231/13_Summer_2021_Q2) ANSWER

The cubic equation $2x^3 - 4x^2 + 3 = 0$ has roots α, β, γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) State the value of S_1 and find the value of S_2. [3]

(b) **(i)** Express S_{n+3} in terms of S_{n+2} and S_n. [1]

(ii) Hence, or otherwise, find the value of S_4. [2]

(c) Use the substitution $y = S_1 - x$, where S_1 is the numerical value found in part **(a)**, to find and simplify an equation whose roots are $\alpha+\beta, \beta+\gamma, \gamma+\alpha$. [3]

(d) Find the value of $\frac{1}{\alpha+\beta} + \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha}$. [2]

1 - (9231/11_Summer_2020_Q2) QUESTION

(a)	$y = x^2$	**B1**
	$6y^{\frac{3}{2}} + py - 3y^{\frac{1}{2}} - 5 = 0 \Rightarrow y^{\frac{1}{2}}(6y-3) = -py + 5$ $y(6y-3)^2 = (-py+5)^2 \Rightarrow y(36y^2 - 36y + 9) = p^2y^2 - 10py + 25$	**M1**
	$36y^3 - (p^2+36)y^2 + (10p+9)y - 25 = 0$	**A1**
		3
(b)(i)	$\alpha^2 + \beta^2 + \gamma^2 = \frac{p^2+36}{36}$	**B1**
	$\frac{p^2+36}{36} = -\frac{2p}{6} \Rightarrow p^2 + 12p + 36 = 0$	**M1**
	$p = -6$	**A1**
		3
(b)(ii)	$6(\alpha^3 + \beta^3 + \gamma^3) = 6(\alpha^2 + \beta^2 + \gamma^2) + 3(\alpha + \beta + \gamma) + 15$	**M1**
	$\alpha^3 + \beta^3 + \gamma^3 = 5$	**A1**
		2

2 - (9231/12_Summer_2020_Q2) QUESTION

(a)	$y = x^2$	B1
	$6y^{\frac{3}{2}} + py - 3y^{\frac{1}{2}} - 5 = 0 \Rightarrow y^{\frac{1}{2}}(6y-3) = -py+5$ $y(6y-3)^2 = (-py+5)^2 \Rightarrow y(36y^2 - 36y + 9) = p^2y^2 - 10py + 25$	M1
	$36y^3 - (p^2+36)y^2 + (10p+9)y - 25 = 0$	A1
		3
(b)(i)	$\alpha^2 + \beta^2 + \gamma^2 = \frac{p^2+36}{36}$	B1
	$\frac{p^2+36}{36} = -\frac{2p}{6} \Rightarrow p^2 + 12p + 36 = 0$	M1
	$p = -6$	A1
		3
(b)(ii)	$6(\alpha^3 + \beta^3 + \gamma^3) = 6(\alpha^2 + \beta^2 + \gamma^2) + 3(\alpha + \beta + \gamma) + 15$	M1
	$\alpha^3 + \beta^3 + \gamma^3 = 5$	A1
		2

3 - (9231/13_Summer_2020_Q1) QUESTION

(a)	$y = x^{-1}$	**B1**
	$7y^{-3} + 3y^{-2} + 5y^{-1} + 1 = 0 \Rightarrow y^3 + 5y^2 + 3y + 7 = 0$	**M1 A1**
		3
(b)	$\alpha^{-2} + \beta^{-2} + \gamma^{-2} = (-5)^2 - 2(3) = 19$	**M1 A1**
(c)	$\alpha^{-3} + \beta^{-3} + \gamma^{-3} = -5(19) - 3(-5) - 21 = -101$	**M1 A1**
		4

4 - (9231/11_Winter_2020_Q3) QUESTION

(a)	$y = x^3 \Rightarrow x = y^{\frac{1}{3}}$	**B1**	Substitutes.
	$y + cy^{\frac{1}{3}} + 1 = 0 \Rightarrow -c^3 y = (y+1)^3 = y^3 + 3y^2 + 3y + 1$	**M1**	Correct attempt to eliminate cube root.
	$y^3 + 3y^2 + (3 + c^3)y + 1 = 0$	**A1**	
		3	
(b)	$\alpha^3 + \beta^3 + \gamma^3 = -3 \quad \alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = 3 + c^3$	**B1 FT**	Using *their* answer to **(a)**.
	$\alpha^6 + \beta^6 + \gamma^6 = (-3)^2 - 2(3 + c^3)$	**M1**	$\alpha^6 + \beta^6 + \gamma^6 = (\alpha^3 + \beta^3 + \gamma^3)^2 - 2(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3)$
	$= 3 - 2c^3$	**A1**	AG
		3	
(c)	$\alpha^3\beta^3\gamma^3 = -1$	**B1**	If using *their* answer to **(a)** FT
	$\begin{vmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{vmatrix} = 1 - (\alpha^6 + \beta^6 + \gamma^6) + 2\alpha^3\beta^3\gamma^3 = 2c^3 - 4$	**M1 A1**	Evaluates determinant.
	$2c^3 - 4 = 0$	**M1**	Sets determinant equal to zero.
	$c = \sqrt[3]{2}$	**A1**	
		5	

5 - (9231/12_Winter_2020_Q1) QUESTION

(a)	$d=1$	**B1**	
		1	
(b)	$y=x+1 \Rightarrow x=y-1$	**B1**	Uses correct substitution.
	$y^3+(b-3)y^2+(c-2b+3)y+b-c=0$	**M1 A1**	Substitutes and expands.
	Alternative method for question 1(b)		
	$(\alpha+1)(\beta+1)+(\alpha+1)(\gamma+1)+(\beta+1)(\gamma+1)=c-2b+3$	**B1**	
	$(\alpha+1+\beta+1+\gamma+1)=3-b,\ (\alpha+1)(\beta+1)(\gamma+1)=c-b$	**M1**	Using these relationships.
	$y^3+(b-3)y^2+(c-2b+3)y+b-c=0)$	**A1**	
		3	
(c)	$\beta+1=-(b-3)$	**B1**	Uses sum of roots.
	$-(\alpha+1)(\alpha+1)=c-2b+3$	**B1**	Uses sum of products in pairs.
	$-(\alpha+1)(\beta+1)(\alpha+1)=-(b-c)$	**M1**	Applies product of roots.
	$\Rightarrow(c-2b+3)(b-3)=b-c$	**A1**	AG
		4	

6 - (9231/13_Winter_2020_Q3) QUESTION

(a)	$y = x^3 \Rightarrow x = y^{\frac{1}{3}}$	B1	Substitutes.
	$y + cy^{\frac{1}{3}} + 1 = 0 \Rightarrow -c^3 y = (y+1)^3 = y^3 + 3y^2 + 3y + 1$	M1	Correct attempt to eliminate cube root.
	$y^3 + 3y^2 + (3 + c^3)y + 1 = 0$	A1	
		3	
(b)	$\alpha^3 + \beta^3 + \gamma^3 = -3 \quad \alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = 3 + c^3$	B1 FT	Using *their* answer to **(a)**.
	$\alpha^6 + \beta^6 + \gamma^6 = (-3)^2 - 2(3 + c^3)$	M1	$\alpha^6 + \beta^6 + \gamma^6 = (\alpha^3 + \beta^3 + \gamma^3)^2 - 2(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3)$
	$= 3 - 2c^3$	A1	AG
		3	
(c)	$\alpha^3\beta^3\gamma^3 = -1$	B1	If using *their* answer to **(a)** FT
	$\begin{vmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{vmatrix} = 1 - (\alpha^6 + \beta^6 + \gamma^6) + 2\alpha^3\beta^3\gamma^3 = 2c^3 - 4$	M1 A1	Evaluates determinant.
	$2c^3 - 4 = 0$	M1	Sets determinant equal to zero.
	$c = \sqrt[3]{2}$	A1	
		5	

7 - (9231/11_Summer_2021_Q3) QUESTION

(a)	$y = x^3$	**B1**	Correct substitution.
	$y^{\frac{4}{3}} - 2y - 1 = 0 \Rightarrow y^4 = (2y+1)^3 = 8y^3 + 12y^2 + 6y + 1$	**M1**	Obtains an equation not involving radicals.
	$y^4 - 8y^3 - 12y^2 - 6y - 1 = 0$	**A1**	
	$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = 8$	**B1 FT**	
		4	
(b)	$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3} = \frac{\alpha^3\beta^3\delta^3 + \alpha^3\beta^3\gamma^3 + \beta^3\gamma^3\delta^3 + \alpha^3\gamma^3\delta^3}{\alpha^3\beta^3\gamma^3\delta^3} = \frac{6}{-1}$	**M1** **A1 FT**	Relates $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3}$ to coefficients.
	–6	**A1**	
		3	
(c)	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 2(\alpha^3 + \beta^3 + \gamma^3 + \delta^3) + 4$	**M1**	Uses original equation.
	= 20	**A1**	
		2	

8 - (9231/12_Summer_2021_Q3) QUESTION

(a)	$y = x^3$	**B1**	Correct substitution.
	$y^{\frac{4}{3}} - 2y - 1 = 0 \Rightarrow y^4 = (2y+1)^3 = 8y^3 + 12y^2 + 6y + 1$	**M1**	Obtains an equation not involving radicals.
	$y^4 - 8y^3 - 12y^2 - 6y - 1 = 0$	**A1**	
	$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = 8$	**B1 FT**	
		4	
(b)	$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3} = \frac{\alpha^3\beta^3\delta^3 + \alpha^3\beta^3\gamma^3 + \beta^3\gamma^3\delta^3 + \alpha^3\gamma^3\delta^3}{\alpha^3\beta^3\gamma^3\delta^3} = \frac{6}{-1}$	**M1** **A1 FT**	Relates $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3}$ to coefficients.
	-6	**A1**	
		3	
(c)	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 2(\alpha^3 + \beta^3 + \gamma^3 + \delta^3) + 4$	**M1**	Uses original equation.
	$= 20$	**A1**	
		2	